**FINAL: ALGEBRA II**

Date: **4th May 2018**
The Total points is **110** and the maximum you can score is **100** points.

Notation: $\mathbb{F}_p$ denotes the finite field of $p$ elements where $p$ is a prime number.

(1) (8+8+8+8=32) Mark all correct options.
  (a) Which of the following fields are a splitting field of $X^8 - 5$ over $\mathbb{Q}(i)$?
    (i) $\mathbb{Q}(i)(\sqrt[8]{5})$
    (ii) $\mathbb{Q}(i)(\sqrt[8]{5}, e^{2\pi i/8})$
    (iii) $\mathbb{Q}(i)(\sqrt[8]{5}, \sqrt{2})$
    (iv) $\mathbb{Q}(i)(\sqrt[11]{3}, \sqrt{3})$
  (b) Which of the following statements are true?
    (i) Every algebraic extension of $\mathbb{Q}(x)$ is separable.
    (ii) Every algebraic extension of $\mathbb{F}_p(x)$ is separable. Here $x$ is an indeterminate.
    (iii) Every algebraic extension of $\mathbb{F}_p$ is separable.
    (iv) Every Galois extension of $\mathbb{F}_p(x_1, x_2)$ is separable. Here $x_1$ and $x_2$ are indeterminates.
  (c) Which of the following field extensions are purely transcendental?
    (i) $\mathbb{F}_p(x_1, x_2)/\mathbb{F}_p$ where $x_1$ and $x_2$ are indeterminates.
    (ii) $\mathbb{Q}(\sqrt[3]{7})/\mathbb{Q}$
    (iii) $\mathbb{C}(x)/\mathbb{C}(x^5)$
    (iv) $\mathbb{C}(x)[\sqrt{x}]/\mathbb{C}$
  (d) Let $G$ and $H$ be profinite groups. Which of the following are true?
    (i) $G \times H$ is a profinite group.
    (ii) Every finite index closed subgroup of $G$ is also open in $G$.
    (iii) Every subgroup of $G$ is closed in $G$.
    (iv) Every nonempty open subset of $G$ is a subgroup of $G$.

(2) (8 points) Give an example of the following (without justification).
  (a) A finite extension which is not simple.
  (b) A separable field extension which is not Galois.
  (c) An inseparable extension which is not purely inseparable.
  (d) An algebraic extension of $\mathbb{Q}$ which is not solvable by radicals.

(3) (5+10=15 points) Let $\Omega/F$ be a field extension. When are two field extensions of $F$ contained in $\Omega$ called linearly disjoint? Let $K/F$ and $L/F$ be linearly disjoint Galois extensions contained in $\Omega$ with Galois group $G_1$ and $G_2$. Compute the Galois group $\mathrm{Gal}(KL/F)$.

(4) (5+10=15 points) Let $F$ be a field of characteristic $p > 0$. When is a finite extension $K/F$ called purely inseparable? Show that $K/F$ is purely inseparable iff there is a unique $F$-embedding of $K$ into the algebraic closure of $F$.

(5) (8+12=20 points) Let $L/F$ be a finite field extension. Define the norm and the trace map for the extension $L/F$ from $L$ to $F$. Show that if $L/F$ is a Galois extension then there exists an $\alpha \in L$ such that its trace $T_{L/F}(\alpha) \neq 0$.

(6) (10+10=20 points) Let $F$ be a field of characteristic $p$ and $a \in F$ be such that the polynomial $f(Z) = Z^p - Z - a$ is irreducible in $F[Z]$. Let $b \in F$ and $\alpha$ be a root of $f(Z)$ in some fixed algebraic closure $\bar{F}$ of $F$. Compute the minimal polynomial of $\alpha + b$ over $F$. Let $\beta \in \bar{F}$ be such that $c := \beta^p - \beta \in F$. Show that if $a - uc = x^p - x$ for some nonzero $u \in \mathbb{F}_p$ and some $x \in F$, then $F[\alpha] = F[\beta]$.